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LETTER TO THE EDITOR

A note on recent Lie parasuperstructures

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Abstract. We discuss two recently published triple products seen as generalizations in parasupersymmetric quantum mechanics of the usual anticommutator between odd operators in the supersymmetric context. They lead to Lie parasuperalgebras which are not isomorphic in general.

We have recently pointed out remarkable Lie structures (Beckers and Debergh 1990a) that have been called for evident reasons 'Lie parasuperalgebras' and contain Lie superalgebras as particular cases. In fact, these mathematical results were motivated by physical properties (Beckers and Debergh 1990b) that we have exploited in parasupersymmetric quantum mechanics, *slightly* modified with respect to the original Rubakov–Spiridonov proposal (Rubakov and Spiridonov 1988). We learned in June (Beckers and Debergh 1990c) that the *same* expression, 'Lie parasuperalgebra', has been used by Durand and Vinet (1990) in connection with structures suggested by the *exact* context Rubakov and Spiridonov proposed (1988).

In this letter we want to show some of the differences and analogies contain in both contributions (Beckers and Debergh 1990a, Durand and Vinet 1990) and to discuss the implied triple products.

Let us remember that the typical *trilinear* relations of the Rubakov–Spiridonov algebra (Rubakov and Spiridonov 1988) on the charges and the Hamiltonian are

$$Q^2Q^\dagger + QQ^\dagger Q + Q^\dagger Q^2 = 4QH_{\mathrm{RS}} \qquad Q^{\dagger 2}Q + Q^\dagger QQ^\dagger + QQ^{\dagger 2} = 4Q^\dagger H_{\mathrm{RS}} \tag{1}$$

which can always be written in the form

$$\begin{aligned} &Q\{Q^\dagger, Q\} + Q\{QQ^\dagger\} + Q^\dagger\{QQ\} = 8QH_{\mathrm{RS}} \\ &Q^\dagger\{QQ^\dagger\} + Q^\dagger\{Q^\dagger, Q\} + Q\{Q^\dagger, Q^\dagger\} = 8Q^\dagger H_{\mathrm{RS}}. \end{aligned} \tag{2}$$

Here we refer to the Rubakov–Spiridonov Hamiltonian

$$H_{\mathrm{RS}} = \tfrac{1}{2}p^2 + \tfrac{1}{4}\begin{pmatrix} 3W_1' + W_2' + W_1^2 + W_2^2 & 0 & 0 \\ 0 & W_2' - W_1' + W_1^2 + W_2^2 & 0 \\ 0 & 0 & W_1^2 + W_2^2 - W_1' - 3W_2' \end{pmatrix} \tag{3}$$

expressed in terms of the two superpotentials $W_1(x)$ and $W_2(x)$ (along with their first derivatives with respect to x) appearing in the parasupercharge Q (and $Q^\dagger$) given by

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ p + \mathrm{i}W_1(x) & 0 & 0 \\ 0 & p + \mathrm{i}W_2(x) & 0 \end{pmatrix} \qquad p \equiv -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}. \tag{4}$$

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Such a context implies the following constraint (Rubakov and Spiridonov 1988) on the superpotentials

$$(W_2^2 - W_1^2)' + W_2'' + W_1'' = 0 \qquad W_1 + W_2 \neq 0$$

or

$$W_2^2 - W_1^2 + W_2' + W_1' = c = \text{constant} \neq 0. \tag{5}$$

Then, if one defines the triple product

$$\{A, B, C\} = A\{B, C\} + C\{A, B\} + B\{C, A\} \tag{6}$$

the relations (2) become

$$\{Q, Q^\dagger, Q\} = 8QH_{\mathrm{RS}} \qquad \{Q^\dagger, Q, Q^\dagger\} = 8Q^\dagger H_{\mathrm{RS}}. \tag{7}$$

This is just the origin of the three-linear symmetric product introduced by Durand and Vinet (1990) leading to the definition of their parasuperalgebra.

Let us now point out that we have modified (Beckers and Debergh 1990b) the relations (1) by asking that the following hold:

$$[Q, [Q^\dagger, Q]] = 2QH_{\mathrm{PSS}} \qquad [Q^\dagger, [Q, Q^\dagger]] = 2Q^\dagger H_{\mathrm{PSS}} \tag{8}$$

where our parasuperHamiltonian (Beckers and Debergh 1990b) H_{PSS} is given by

$$H_{\mathrm{PSS}} = \tfrac{1}{2}p^2 + \tfrac{1}{2}\begin{pmatrix} 2W_1^2 - W_2^2 - W_2' & 0 & 0 \\ 0 & 2W_1^2 - W_2^2 - 2W_1' - W_2' & 0 \\ 0 & 0 & 2W_2^2 - W_1^2 + W_1' \end{pmatrix} \tag{9}$$

when the constraint (5) is replaced by

$$W_2^2 - W_1^2 + W_2' + W_1' = 0 \tag{10}$$

admitting in particular the choice $W_1(x) = -W_2(x)$. Here, the relevant triple product appears as a double commutator satisfying identities such that

$$[A, [B, C]] = \{\{A, B\}, C\} - \{\{A, C\}, B\}. \tag{11}$$

It is simply related to the definition (6) by

$$[A, [B, C]] = 2\{A, B, C\} - 2\{B, \{C, A\}\} - \{A, \{B, C\}\}. \tag{12}$$

The two triple products are thus non-equivalent in general but can be connected to each other. The Lie parasuperalgebras displayed in both contributions (Beckers and Debergh 1990a, Durand and Vinet 1990) are thus generally non-equivalent and non-isomorphic structures.

Let us now compare their respective contents by coming back to the typical properties of the common *parafermionic* operators, b and $b^\dagger$, included in the definition of the parasupercharge (4) given in the form

$$Q = \frac{1}{2\sqrt{2}}[(p + \mathrm{i}W_1)b^\dagger b^2 + (p + \mathrm{i}W_2)b^2 b^\dagger]. \tag{13}$$

These parafermionic operators are such that (Ohnuki and Kamefuchi 1982), for arbitrary $n = 0, 1, \ldots, p$, we have

$$b|n+1\rangle = [(n+1)(p-n)]^{1/2}|n\rangle \tag{14a}$$

and

$$b^\dagger|n\rangle = [(n+1)(p-n)]^{1/2}|n+1\rangle. \tag{14b}$$

The first triple product (6), in terms of the following parafermionic operators, then gives

$$\{b, b^{\dagger}, b\}|n\rangle = 2(b^2b^{\dagger} + bb^{\dagger}b + b^{\dagger}b^2)|n\rangle$$
$$= 2(3np - 3n^2 + 3n - 2)b|n\rangle \qquad n = 1, 2, \ldots, p.$$

For $p = 1$ and $n = 1$ we get

$$\{b, b^{\dagger}, b\} = 2b \tag{15a}$$

and for $p = 2$ and $n = 1$ *or* 2, we obtain

$$\{b, b^{\dagger}, b\} = 8b \tag{15b}$$

showing that the first typically parafermionic case ($p = 2$) *and* the fermionic case ($p = 1$)—an *ad hoc* context for supersymmetric developments—are characterized by different relations due to different coefficients in (15). The required inclusion of supersymmetry (Witten 1981) (SS) inside parasupersymmetry (PSS) will then demand different definitions of parasupercharges with respect to supercharges; not only in their matrix forms, but also in their respective coefficients.

The second triple product (11), in terms of the *same* parafermionic operators, can also be evaluated in the above way through (14). We immediately get

$$[b, [b^{\dagger}, b]]|n\rangle = (2bb^{\dagger}b - b^2b^{\dagger} - b^{\dagger}b^2)|n\rangle$$
$$= 2b|n\rangle \qquad \forall n, p$$

so that for $p = 1$ *or* 2, we have

$$[b, [b^{\dagger}, b]] = 2b \tag{16}$$

showing that the inclusion PSS $\supset$ SS is natural here and does not require any adjustment of the definitions of parasupercharges and supercharges.

By comparing results (15) and (16), we once again notice that the two triple products (here on the b's) are not equivalent: the first one (6) varies for different p-values while the second one does not. This induces non-trivial terms in (12) besides the ones containing $[A, [B, C]]$ and $\{A, B, C\}$.

There are two cases ensuring that the Beckers–Debergh (1990a) and Durand–Vinet (1990) parasuperalgebras are isomorphic structures: the $p = 1$-fermionic case (cf (15*a*) and (16)) which is a trivial case reducing the structures to Lie superalgebras, *and* the $p = 2$-parafermionic case *iff* the two parasuperHamiltonians H_{RS} and H_{PSS} coincide as it is only realized with our constraint (10). In fact, we then immediately get

$$\{Q, Q^{\dagger}, Q\} = 4[Q, [Q^{\dagger}, Q]] \tag{17}$$

from (7) and (8), and from (15*b*) and (16)

$$\{b, b^{\dagger}, b\} = 4[b, [b^{\dagger}, b]] = 8b. \tag{18}$$

Let us recall that, in this context, we have obtained (Beckers and Debergh 1990b) the interesting *oscillator-like* operator

$$H_{\mathrm{PSS}} = \frac{1}{2}(p^2 + \omega^2 x^2) + \frac{\omega}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{19}$$

leading to a non-degenerate ground-state with zero energy, to the concept of exact parasupersymmetry and to the direct inclusion PSS ⊃ SS. Moreover, such a context gives the remarkable possibility to extend (Beckers and Debergh 1990c) the standard procedure (Witten 1981) to the spin-orbit coupling procedure (Ui 1984, Ui and Takeda 1984, Balantekin 1985). These properties are thus typically associated with our triple product defined in (8) while the Durand-Vinet considerations ask for *re*definitions of the charges in order to ensure the inclusion PSS ⊃ SS do not contain the zero energy value and are not convenient for an extension to the spin-orbit coupling procedure.

Let us end this letter by noticing the existence of another triple product—a third one—already introduced in connection with parastatistics. It has been defined and used by Ohnuki and Kamefuchi (1982) in realizations of paracommutation relations. Let us denote this Ohnuki-Kamefuchi triple product by

$$\langle A, B, C\rangle_+ = ABC + CBA \tag{20}$$

so that it is simply related to ours, given by (11), as follows:

$$[A, [B, C]] = \langle A, B, C\rangle_+ - \langle B, C, A\rangle_+ . \tag{21}$$

Applied to our states $|n\rangle$, this third triple product (20), expressed in terms of the previous parafermionic operators, leads to

$$\langle b, b^\dagger, b\rangle_+ |n\rangle = 2n(p - n + 1)b|n\rangle. \tag{22}$$

For $p = 1$ and $n = 1$, we get

$$\langle b, b^\dagger, b\rangle_+ = 2b \tag{23a}$$

and for $p = 2$ and $n = 1$ *or* 2 we obtain

$$\langle b, b^\dagger, b\rangle_+ = 4b. \tag{23b}$$

If once again this triple product coincides with the two preceding ones when $p = 1$, i.e. in the supersymmetric context, we also note that the $p = 2$-parasupersymmetric case leads to the relations

$$\langle b, b^\dagger, b\rangle_+ = 2[b, [b^\dagger, b]] = \tfrac{1}{2}\{b, b^\dagger, b\} \tag{24}$$

showing that the inclusion PSS ⊃ SS would for example also ask for redefinitions of supercharges. Among the *three* triple products we thus conclude on the advantages shown by ours, defined in (8) on the parasupercharges or in (16) on the parafermionic creation and annihilation operators.

As a last comment we want to draw the attention of the reader to a link we noticed between the elements discussed here and ternary algebras (Bars and Gunaydin 1979) characterized by triple products seen as mappings of a vector space V, in the form $V \otimes V \otimes V \to V$. In particular the triple products defined above on the parafermionic operators correspond to Jordan superternary algebras, showing that the latter also have an important role to play in physical theories now related with parastatistics and supersymmetry, or developed in parasupersymmetric quantum mechanics. As an example let us point out that we need the condition

$$x \circ y \circ z = z \circ y \circ x$$

for a Jordan superternary algebra which can be realized with

$$x \circ y \circ z = xyz + zyx = \langle x, y, z\rangle_+$$

according to our notation (20).

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